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Linear Perturbations of Gauge Fields

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It is shown that under certain weak conditions (the vanishing of the field strength along a family of self-dual or anti-self-dual geodesic two-surfaces), in a curved or flat space-time, the linear perturbations of a given gauge field configuration can be expressed in terms of the solutions of a single second-order linear partial differential equation for a matrix potential. The particular case of the self-dual gauge fields is treated in some detail.

1. INTRODUCTION

The fact that the Yang-Mills equations, which govern the gauge fields, like Einstein's field equations, are a coupled system of nonlinear partial differential equations is an obstacle to the analysis of the models of the fundamental interactions based on those fields. Hence, even though gauge fields interact with quantum matter and they must be quantized too, the study of the solutions of the classical Yang-Mills equations represents a valuable step in the understanding of the theory.

Gauge fields are, from the geometrical point of view, similar to the gravitational field in the general theory of relativity. In both cases, by using the complex extension of space-time, it has been possible, under certain restrictions which have a well-defined geometrical meaning, to obtain many nontrivial results concerning restricted classes of solutions and to construct explicitly examples of such fields.

For instance, the self-dual gauge fields, or the self-dual space-times, can be characterized by the fact that the field strength, or the curvature, respectively, vanishes along all the anti-self-dual surface elements, which

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are totally null and necessarily complex. In a flat or self-dual space-time, the anti-self-dual surface elements form families, or congruences, of complex two-dimensional surfaces which are geodesic. Due to the existence of such surfaces the field equations can be reduced or translated to another mathematical structure [see, e.g., Yang (1977) and Ward (1977) for the case of gauge fields and Plebański (1975) and Penrose and Ward (1980) for the gravitational case].

A more general class consists of those fields such that the field strength, or the curvature, vanishes only along a single congruence of two-dimensional anti-self-dual geodesic surfaces. For this class, the field equations without sources reduce to a single nonlinear differential equation [Torres del Castillo (1985) in the case of gauge fields and Plebański and Robinson (1976), Finley and Plebański (1976), and Torres del Castillo (1983) in the gravitational case]; however, due to the necessity of employing complex coordinates and to the nonlinearity of the field equations, the problem remains of determining which solutions of these equations correspond to real fields. On the other hand, the (linear) perturbations of a given solution of these field equations do obey linear equations. Therefore, the real and imaginary parts of a complex perturbation represent real perturbations and, in this case, the formalism based on the complex extension of the space-time can be applied directly. In fact, in this way it is found that the perturbations of the algebraically special solutions of the Einstein vacuum field equations are determined by a single linear second-order differential equation for a complex potential function (Torres del Castillo, 1986)—a result previously obtained by other means (Chrzanowski, 1975; Wald, 1978; Kegeles and Cohen, 1979).

In this paper, using the results given in Torres del Castillo (1985a), we show that, for an arbitrary gauge group, the perturbations of a solution of the source-free Yang-Mills equations such that its field strength vanishes along a congruence of two-dimensional totally null surfaces, suitably correlated with the curvature of the space-time, are given locally in terms of a matrix potential which fulfills a linear second-order differential equation. In the special case of the self-dual gauge fields in a flat or self-dual space-time, we reformulate some results previously given which relate the linear perturbations with the solutions of a linear system whose integrability conditions amount to the self-duality of the background gauge field. In the derivation presented here we make use of complex coordinates induced by the totally null surfaces mentioned above and of null tetrads associated with them. However, the final results are given with respect to an arbitrary null tetrad. The spinor notation is used throughout this paper. [An excellent presentation of the spinor formalism can be found in, e.g., Pirani (1965). See also Plebański (1975).]

2. THE REDUCED FORM OF THE YANG–MILLS EQUATIONS

In terms of the spinor notation, a spinor $l_{\dot{A}}$ defines, at each point of the space–time where it does not vanish, a totally null subspace formed by the vectors of the form $m_B l_{\dot{A}}$, with m_B arbitrary. With the appropriate convention for the duality operation, the bivector corresponding to these subspaces, whose spinor components are $l_{\dot{A}} l_{\dot{B}}$, is anti-self-dual. The condition for these subspaces to be tangent to a family of two-dimensional surfaces is

$$l^{\dot{A}} l^{\dot{B}} \nabla_{C\dot{A}} l_{\dot{B}} = 0 \tag{1}$$

If equation (1) is satisfied and additionally

$$l^{\dot{A}} l^{\dot{B}} l^{\dot{C}} C_{\dot{A}\dot{B}\dot{C}\dot{D}} = 0 \tag{2}$$

where $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ denotes the spinor components of the anti-self-dual part of the conformal curvature (Weyl spinor), then there exist (complex) coordinates $q^A, p^A (A = 1, 2)$ in terms of which the space–time metric takes the form

$$g = 2\phi^{-2}\, dq^A \underset{S}{\otimes} (dp_A + Q_{AB}\, dq^B) \tag{3}$$

where ϕ and Q_{AB} are complex functions, with $Q_{AB} = Q_{BA}$. The function ϕ satisfies the condition

$$l^{\dot{A}} \nabla_{B\dot{C}}\, l_{\dot{A}} = l_{\dot{C}}\, l^{\dot{A}} \partial_{B\dot{A}} \ln \phi \tag{4}$$

(Torres del Castillo, 1983, 1984). For a space–time that satisfies the Einstein vacuum field equations, each of the equations (1) and (2) implies the other. By virtue of (2) the space–time is said to be algebraically special.

The two-dimensional surfaces defined by $l_{\dot{A}}$ are given by $dq^A = 0$ and the tangent vectors to them are spanned by $\partial/\partial p^A$. From equation (3) it is clear that the metric vanishes on these surface, i.e., they are totally null and, as a consequence of (1), geodesic.

As is shown in Torres del Castillo (1985), any solution of the source-free Yang–Mills equations such that

$$l^{\dot{A}} l^{\dot{B}} F_{\dot{A}\dot{B}} = 0 \tag{5}$$

where $F_{\dot{A}\dot{B}}$ denotes the spinor components of the anti-self-dual part of the field strength and where $l_{\dot{A}}$ satisfies equations (1) and (2), is given, in an arbitrary gauge, by

$$A = A_\mu\, dx^\mu = M^{-1}\, dM - M^{-1}(\varepsilon p_A - \partial H/\partial p^A) M\, dq^A \tag{6}$$

where ε is an arbitrary matrix, which depends on q^R only, M is a gauge-dependent matrix whose presence is necessary in order for A to take values in the Lie algebra of the gauge group along all real directions in the space–time, and H is a potential matrix, which must fulfill the equation

$$D^A \partial_A H + \partial^A H \, \partial_A H + [p^A \partial_A H - H, \varepsilon] = \nu \tag{7}$$

where ν is an arbitrary matrix, which depends on q^R only, and where we have used the abbreviations

$$\partial_A = \partial/\partial p^A, \qquad D_A = \partial/\partial q^A + Q_{AB}\partial^B \tag{8}$$

[all the spinor indices are raised and lowered according to the convention $\psi_A = \varepsilon_{AB}\psi^B$, $\psi^B = \varepsilon^{AB}\psi_A$, and similarly for dotted indices].

The coordinates q^A, p^A induce the complex null tetrad

$$\partial_{A\dot{1}} = \sqrt{2}\partial_A, \qquad \partial_{A\dot{2}} = \sqrt{2}\phi^2 D_A \tag{9}$$

which satisfies $\partial_{A\dot{B}} \cdot \partial_{C\dot{D}} = -2\varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}$. With respect to this tetrad the components of the potential A given in (6) are

$$\begin{aligned} A_{B\dot{1}} &= \sqrt{2}M^{-1}\partial_B M \\ A_{B\dot{2}} &= \sqrt{2}\phi^2\{M^{-1}D_B M - M^{-1}(\varepsilon p_B - \partial_B H)M\} \end{aligned} \tag{10}$$

while the field strength components are given by

$$\begin{aligned} F_{\dot{1}\dot{1}} &= 0, \qquad F_{\dot{1}\dot{2}} = 2\phi^2 M^{-1}\varepsilon M \\ F_{\dot{2}\dot{2}} &= 2\phi^4 M^{-1}\{p^A \partial\varepsilon/\partial q^A + [H, \varepsilon] + \nu\}M \\ F_{AB} &= 2\phi^2 M^{-1}(\partial_A\partial_B H)M \end{aligned} \tag{11}$$

Condition (5) [which in the tetrad (9) amounts to $F_{\dot{1}\dot{1}} = 0$] means that for any pair of vectors tangent to the two-dimensional surfaces defined by $l_{\dot{A}}$, v^μ, and w^μ, $F_{\mu\nu}v^\mu w^\nu = 0$, or equivalently, that the connection defined by A restricted to these surfaces is flat. For a real field, condition (5) implies $l^A l^B F_{AB} = 0$, where F_{AB} corresponds to the self-dual part of the field strength, and conversely. However, the self-dual part of the field strength generated by a solution of (7) may not satisfy a condition of the form $l^A l^B F_{AB} = 0$, but it can be more general. For instance, by taking $\varepsilon = \nu = 0$ one gets $F_{\dot{A}\dot{B}} = 0$, while F_{AB} is completely arbitrary and the solutions of (7) yield all the self-dual gauge fields in a space–time that admits a solution of (1) and (2).

3. LINEAR PERTURBATIONS

Assuming now that H is a solution of equation (7) corresponding to a given real solution of the source-free Yang–Mills equations and that $H + \delta H$ also satisfies equation (7) to first order in δH, one obtains, with ε and ν fixed, that

$$D^A\partial_A \,\delta H + [\partial^A H - \varepsilon p^A, \partial_A \,\delta H] + [\varepsilon, \delta H] = 0$$

By using equations (9)–(11) this equation for δH can be written in the following gauge-covariant form, valid only for the tetrad (9):

$$\partial_{\dot 2}^{A}\{\partial_{A\dot 1}\chi+[A_{A\dot 1},\chi]\}+[A_{\dot 2}^{A},\partial_{A\dot 1}\chi+[A_{A\dot 1},\chi]]+[F_{\dot 1\dot 2},\chi]=0 \qquad (12)$$

where $\chi\equiv M^{-1}(\delta H)M$. Then, from equations (6) and (9)–(11) it follows that the perturbations of the field expressed in the basis (9) are

$$\begin{aligned}&\delta A=\frac{1}{\sqrt{2}}\{\partial_{A\dot 1}\chi+[A_{A\dot 1},\chi]\}\,dq^{A}\\ &\delta F_{\dot 1\dot 1}=\delta F_{\dot 1\dot 2}=0,\qquad \delta F_{\dot 2\dot 2}=-\phi^{2}[F_{\dot 1\dot 2},\chi]\\ &\delta F_{AB}=\phi^{2}\{\partial_{A\dot 1}(\partial_{B\dot 1}\chi+[A_{B\dot 1},\chi])+[A_{A\dot 1},\partial_{B\dot 1}\chi+[A_{B\dot 1},\chi]]\}\end{aligned} \qquad (13)$$

Equations (12) and (13) can be written in a fully covariant form, valid in any gauge and any null tetrad. These expressions, written in a compact and manifestly fully covariant form, are

$$\nabla\!\!\!\!/_{C(\dot A}\phi^{-2}\nabla\!\!\!\!/^{C\dot D}\phi^{2}\psi l_{\dot B)}l_{\dot D}-[F_{(\dot A}^{\dot D},\psi l_{\dot B)}l_{\dot D}]=0 \qquad (14)$$

and

$$\begin{aligned}&\delta A_{B\dot C}=\phi^{-2}\nabla\!\!\!\!/_{B}^{\dot D}(\phi^{2}\psi l_{\dot C}l_{\dot D})\\ &\delta F_{\dot A\dot B}=-[F_{(\dot A}^{\dot D},\psi l_{\dot B)}l_{\dot D}]\\ &\delta F_{AB}=\nabla\!\!\!\!/_{(A}^{\dot C}\phi^{-2}\nabla\!\!\!\!/_{B)}^{\dot D}(\phi^{2}\psi l_{\dot C}l_{\dot D})\end{aligned} \qquad (15)$$

where ψ is a potential matrix, which takes the place of χ [$\psi=\phi^{-2}\Delta^{-1}\chi$, in the notation of Torres del Castillo (1984)] and $\nabla\!\!\!\!/$ denotes the covariant derivative with respect to both the Levi-Civita connection and the gauge field: $\nabla\!\!\!\!/_{A\dot B}=\nabla_{A\dot B}+[A_{A\dot B},\ ]$. These expressions can be obtained from (12) and (13) by means of the procedure given in Torres del Castillo (1984), or it can be verified that equations (14) and (15), when expressed in the tetrad (9) with $l_{\dot A}=\delta_{\dot A}^{\dot 2}$, reduce to (12) and (13). Due to equations (4) and (5), the left-hand side of (14) is proportional to $l_{\dot A}l_{\dot B}$. Therefore, equation (14) constitutes just one differential condition on ψ, equivalent to (12). The combination $\psi l_{\dot C}l_{\dot D}$, which appears in (14) and (15), plays the role of a matrix Hertz potential.

Equations (14) and (15) are very similar to those found for the gravitational perturbations of the algebraically special solutions of the Einstein vacuum field equations (Kegeles and Cohen, 1979; Torres del Castillo, 1986) and in the case where the fields are 1×1 matrices they reduce to (the complex conjugates of) the ones obtained for the electromagnetic perturbations (Wald, 1978; Kegeles and Cohen, 1979; Torres del Castillo, 1984, 1985); in fact, except for the terms containing $F_{\dot A\dot B}$, equations (14) and (15) can be obtained from (the complex conjugates of) the corresponding equations for the electromagnetic perturbations, given explicitly in Wald (1978) and Kegeles and Cohen (1979), by replacing the partial derivatives $\partial/\partial x^\mu$ by $\partial/\partial x^\mu + [A_\mu, \;]$ ("minimal coupling rule").

The perturbations given by (15), in general, are not real. However, since we are considering linear perturbations, one can combine the solution given in (15) with its Hermitian conjugate in order to get real fields. If, for instance, the gauge group consists of unitary matrices, then the potential and the field strength must be skew-Hermitian matrices, which, in a null tetrad such that $\overline{\partial_{A\dot{B}}} = \partial_{B\dot{A}}$, amounts to $(A_{B\dot{C}})^\dagger = -A_{C\dot{B}}$ and $(F_{AB})^\dagger = -F_{\dot{A}\dot{B}}$, where the dagger denotes the Hermitian adjoint. Thus, in that case,

$$\delta A_{B\dot{C}} = \phi^{-2}\nabla^{\dot{D}}_{B}(\phi^2 \psi l_{\dot{C}} l_{\dot{D}}) - \bar{\phi}^{-2}\nabla^{D}_{\dot{C}}(\bar{\phi}^2 \psi^\dagger l_B l_D) \tag{16}$$

where $l_B = \overline{l_{\dot{B}}}$, represents a real perturbation, with ψ being a solution of (14). If ψ is replaced by $i\psi$ in (16), another real perturbation is obtained, which can be regarded as derived from (16) by a duality rotation (cf. Chrzanowski, 1975; Wald, 1978), which makes sense in the linear approximation. In general, the pertubation given by (16) will not satisfy the restriction (5) or any other of that form. Instead, it may be completely general, due to the superposition of the fields δF_{AB} and $\delta F_{\dot{A}\dot{B}}$ given in (15).

4. PERTURBATIONS OF SELF-DUAL FIELDS

In a flat or self-dual space–time, a self-dual gauge field can be characterized by the fact that, for every covariantly constant spinor $\pi_{\dot{A}}$, the linear system

$$\pi^{\dot{D}}(\partial_{C\dot{D}}\Xi + A_{C\dot{D}}\Xi) = 0 \tag{17}$$

where Ξ is a matrix that depends parametrically on $\pi_{\dot{A}}$, is integrable (Ward, 1977; Belavin and Zakharov, 1978; Chau Wang et al., 1981; Torres del Castillo, 1985). Similarly, the self-duality of the gauge field $A_{C\dot{D}}$ implies the integrability of

$$\pi^{\dot{D}}\nabla_{C\dot{D}}\psi \equiv \pi^{\dot{D}}(\partial_{C\dot{D}}\psi + [A_{C\dot{D}}, \psi]) = 0 \tag{18}$$

as can be seen by applying $\pi^{\dot{B}}\nabla^{C}_{\dot{B}}$ to equation (18), using the commutation relation

$$\nabla_{C(\dot{A}}\nabla^{C}_{\dot{B})}\psi = -[F_{\dot{A}\dot{B}}, \psi] \tag{19}$$

In fact, if Ξ satisfies (17) and T is a constant matrix, then $\psi = \Xi T \Xi^{-1}$ is a solution of equation (18).

Solutions of (18) also satisfy equation (14), i.e., each solution of equation (18) that can be constructed from those of equation (17) generates a linear perturbation of the given self-dual gauge field. [See Chau Wang *et al.* (1982) for the case of flat space-time.] Indeed, if $l_{\dot{A}}$ and $m_{\dot{A}}$ are covariantly constant spinors such that $m^{\dot{A}} l_{\dot{A}} = 1$, then

$$m^{\dot{A}} l_{\dot{B}} - m_{\dot{B}} l^{\dot{A}} = \delta_{\dot{B}}^{\dot{A}}$$

and from equation (18) we have

$$\pi^{\dot{N}} m_{\dot{N}} l^{\dot{D}} \nabla_{C\dot{D}} \psi = \pi^{\dot{N}} l_{\dot{N}} m^{\dot{D}} \nabla_{C\dot{D}} \psi$$

Since $l_{\dot{A}}$ is assumed to be covariantly constant, from equation (4) it follows that ϕ can be taken equal to one. Thus, taking into account that $F_{\dot{A}\dot{B}} = 0$, the only nontrivial component of the left-hand side of equation (14), obtained by contracting it with $m^{\dot{A}} m^{\dot{B}}$, is $m^{\dot{A}} \nabla^{C\dot{D}} \psi l_{\dot{D}}$, which is equal to $(\pi^{\dot{M}} l_{\dot{M}} / \pi^{\dot{N}} m_{\dot{N}}) m^{\dot{A}} \nabla_{C\dot{A}} \nabla^{C\dot{D}} \psi m_{\dot{D}}$ and vanishes by virtue of (19).

5. CONCLUDING REMARKS

The results presented here show a great similarity with those obtained in the gravitational case, where the study of the linear perturbations has been used in the analysis of the stability of certain solutions.

An essential difference between the Yang–Mills fields and the gravitational field in Einstein's theory of gravity is that in the latter the connection is not the fundamental object of the field, but it is assumed to be determined by the metric of the space-time. This fact implies that, for certain restricted classes of solutions, while the gravitational perturbances are derived from a scalar potential, those of the Yang–Mills fields require a matrix potential.

As has been pointed out by Trautman (1981), if one considers the connections on the bundle of linear frames, the introduction of a metric in the base manifold corresponds to a symmetry-breaking analogous to that produced by a Higgs field in the case of the Yang–Mills fields. Thus, it is to be expected that, with a suitable Higgs field, the symmetry-breaking corresponds to a reduction of the matrix potentials of the gauge fields or of their perturbations.

An interesting problem, not considered here, is that of the inhomogeneous perturbations, where the perturbations of the fields would be related with the sources.

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